The Determination of an Excess Capacitance

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***Abstract*—In the evaluation of "fringing capacitances" one is required to map the specified geometry in the z plane onto the upper half t plane and then to determine the limiting value of the capacitance between the two segments of the real axis—corresponding to the two conductors in the z plane—when one or both of the gaps between them approaches zero. The usual procedure of mapping the upper half t plane onto an infinite parallel plate configuration, which is often more involved than the first mapping, can be eliminated if one recognizes that the capacitance obtained by mapping the upper half t plane onto a rectangle by means of a well-known elliptic function exceeds, in the limit, the correct value by $(\log 2)/\pi$, for each gap involved.**

Consider the problem of determining the geometrical capacitance[1] between the two strips in the t plane of Fig. 1, in the limit as $\delta \to 0$, subject, first, to the condition that the small semicircle is a magnetic wall and then to the condition that the portion of the real axis between the strips is a magnetic wall. The difference between these capacitances will be called the excess capacitance and denoted by C_{ex}. It is equal to $(\log 2)/\pi$. It is independent of the widths of the strips. The transformation

$$s = \int_b^t \frac{(ab)^{1/2}\,dt}{t[(a+t)(b-t)]^{1/2}} \tag{1}$$

maps the upper half of the t plane into the infinite parallel plate structure in the s plane so that the magnetic wall BC in the s plane maps into a small semicircle centered at the origin in the t plane, for sufficiently distant BC. At the same time, the magnetic wall DA in the s plane maps onto the real axis in the t plane. Upon integration,

$$s = -\log\left\{\frac{2(ab)^{1/2}[(a+t)(b-t)]^{1/2} + 2ab}{(b+a)t} + \frac{b-a}{b+a}\right\}. \tag{2}$$

For small δ, the capacitance of the strips in the t plane is given by the capacitance of the parallel plate configuration in the s plane, namely, $C = -s(\delta)/\pi$ or

$$C = \frac{1}{\pi}\left\{\log\frac{4ab}{b+a} - \log\delta\right\} \tag{3}$$

for sufficiently small δ.

If, on the other hand, the line segment between B and C in the t plane is a magnetic wall, the transformation $t = sn^2 u$ maps the interior of the rectangle in the u plane onto the upper half of the t plane so that the line segments joining BC and DA are both magnetic walls. Then it is well known [1] that the capacitance of the two strips in the t plane is given by K/K' when $k^2 = (D-C)(B-A)/(D-B)(C-A)$.[2] In this case, $A = a$, $B = -\delta$, $C = \delta$, and $D = b$, so that

$$k^2 = \frac{(b-\delta)(a-\delta)}{(b+\delta)(a+\delta)} \quad \text{and} \quad k'^2 = \frac{2(a+b)\delta}{ab + (a+b)\delta + \delta^2}. \tag{4}$$

Then, for sufficiently small δ

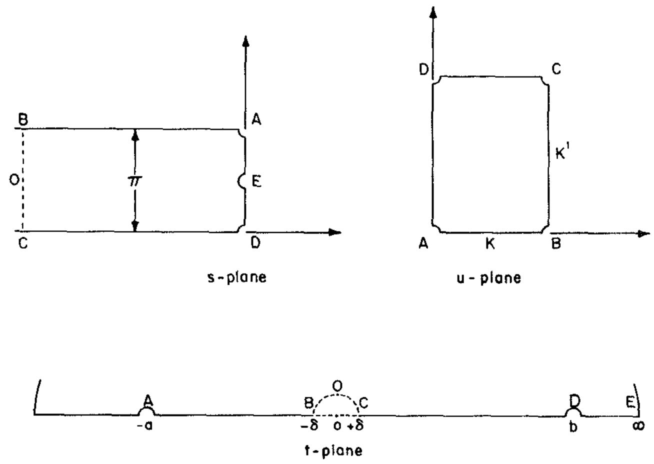

Fig. 1. Coordinate planes of the conformal transformations.

$$K'/K = \frac{1}{\pi}\log\frac{16}{k'^2} = \frac{1}{\pi}\left\{\log\frac{8ab}{a+b} - \log\delta\right\} \tag{5}$$

and finally

$$C_{ex} = K'/K - C = \frac{\log 2}{\pi}. \tag{6}$$

We see that the value of C_{ex} is independent of a and b no matter how small. Thus capacitance between the line segments arising from the lines of force falling inside of the magnetic wall BDC are completely shielded from the external geometry of the figure in the limit as $\delta \to 0$. The same value of excess capacitance will serve if one of the gaps, which approaches zero, occurs at $t = \infty$, a situation which is found in Chen [2, fig. 13]. To show this one need only subject the t plane to an inversion. This interchanges the role of zero and infinity without altering any of the corresponding values of capacitance.

As an example, consider the odd-mode fringing capacitance as determined by the Getsinger [3]. Referring to Bowman [1, p. 83, fig. 4], we require the capacitance of AN with respect to BN. First we determine K'/K having found $k^2 = (D-C)(B-A)/(D-B)(C-A)$ where $A = -\nu - \delta$, $B = -\nu + \delta$, $C = 1$, and $D = 1/k^2$. Thus

$$k^2 = \frac{2\delta(1/k^2 - 1)}{(1/k^2 + \nu - \delta)(1 + \nu + \delta)}. \tag{7}$$

Then, for sufficiently small δ

$$\frac{K'}{K} = \frac{1}{\pi}\log\frac{8(1+\nu)(1+k^2\nu)}{1-k^2} - \frac{1}{\pi}\log\delta. \tag{8}$$

and the total odd-mode capacitance of the system C_0 is given by

$$C_0 = \frac{K'}{K} - \frac{\log 2}{\pi} = \frac{1}{\pi}\log\frac{4(1+\nu)(1+k^2\nu)}{1-k^2} - \frac{1}{\pi}\log\delta \tag{9}$$

where $\nu = cn^2(a,k')/k^2sn^2(a,k')$. This is just the value one would obtain from the more tedious alternate calculation.

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[1] Following the usual convention, Fig. 1 represents a cross section of a geometry which extends infinitely perpendicular to the paper. Thus capacitance means capacitance per unit depth.

[2] It is at this point that the use of C_{ex} simplifies the usual procedure by replacing an integration which is special to the problem under consideration by a single well-known integration.

REFERENCES

[1] F. Bowman, *Introduction to Elliptic Functions with Applications*. New York: Dover, 1961, pp. 56–59.
[2] T.-S. Chen, "Determination of the capacitance, inductance, and characteristic impedance of rectangular lines," *IRE Trans. Microwave Theory Tech.*, vol. MTT-8, pp. 510–519, Sept. 1960.
[3] W. J. Getsinger, "Coupled rectangular bars between parallel plates," *IRE Trans. Microwave Theory Tech.*, vol. MTT-10, pp. 65–72, Jan. 1962.